A Simple Cellular Automaton Model with Limited Braking Rule

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1 Introduction

In 1992, Nagel and Schreckenberg [1] proposed one of the first stochastic cellular automaton (CA) models for analyzing traffic on a one-lane road. Various modifications and extensions such as for the analysis of city traffic or the application to complex road networks have been proposed since then (for reviews see, e.g., [2, 3]). The primary criterion to assess these models is their ability to reproduce empirically observed features of traffic flow [4, 5], which follows from the vehicles' rules of motion.

In the Nagel-Schreckenberg model (NaSch), the road is modeled as a one-dimensional array of sites and each site is occupied by at most one vehicle. The position of the ith car at time t is denoted by x_t^i and the position of its immediate predecessor by x_t^{i+1}. The vehicle's dynamics (i.e., its acceleration and its deceleration) depends on its predecessor on the road. This behaviour is implemented by an update scheme in which the following simple rules are applied to each vehicle in parallel:

1. A car i with speed v_t^i at time t accelerates if the distance to its predecessor $\delta_t^i = x_t^{i+1} - x_t^i$ is large enough. That means if δ_t^i is larger than $v_t^i + 1$ then the speed is advanced by one until the car's maximum speed $v_{\max}$ is reached: $v_{t+1}^i = \min(v_t^i + 1, v_{\max})$.
2. If the distance δ_t^i is too small (i.e., $\delta_t^i \leq v_{t+1}^i$), the car reduces its speed to avoid a collision: $v_{t+1}^i = \min(\delta_t^i - 1, v_{t+1}^i)$.
3. The car's speed is decreased by one at random: $v_{t+1} = \max(0, v_{t+1}^i - 1)$ with probability $1 - p_{\text{acc}}$.
4. Finally, the vehicle moves from its current position x_t^i to its new position $x_{t+1}^i = x_t^i + v_{t+1}^i$.

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Note that a car accelerates by one at maximum (step 1), but it can slow down by more than one (step 2). Step 3 mimics speed fluctuations due to human behaviour. It is essential for traffic jams to occur. Nagel and Schreckenberg focused their analysis on the relationship between the density ρ (the number of cars N divided by the length of the road L) and the traffic flow J (the average number of cars which pass a site per time step). They conclude that the traffic flow increases rapidly up to a certain density above which the average traffic flow decreases as the probability of traffic jams increases rapidly. Despite its simplicity, the model is able to reproduce empirically observed traffic phenomena such as the spontaneous formation of traffic jams. A major drawback of the model is that it allows for unrealistically high deceleration rates (step 2).

With our modification of the NaSch we will present a minimalistic discrete CA model with limited braking capabilities for simulating traffic flow on a single lane. We will also show that this modified model (in the following: mNaSch), unlike the Nagel-Schreckenberg model, tends to converge to steady states. First, let us introduce the modified model.

2 Modified version of the NaSch

Let $L \in \mathbb{N}$ be the number of sites representing the one-lane road. At time t, the car labeled i moves with speed v_t^i which is bounded from above by $\mu(v_{t-1}^{i+1}, \delta_{t-1}^i)$. This upper boundary, whose value depends both on the speed of the leading vehicle and the distance gap, ensures that there are no collisions of two cars as we will show below. In our modified model, a car changes its speed according to the following rule:

$$v_t^i = \begin{cases} v_{t-1}^i + 1 & \text{if } v_{t-1}^i + 1 \leq \mu(v_{t-1}^{i+1}, \delta_{t-1}^i) \text{ and } \xi \leq p_{\text{acc}}, \\ v_{t-1}^i & \text{if } v_{t-1}^i + 1 \leq \mu(v_{t-1}^{i+1}, \delta_{t-1}^i) \text{ and } \xi > p_{\text{acc}}, \\ \mu(v_{t-1}^{i+1}, \delta_{t-1}^i) & \text{otherwise.} \end{cases} \quad (1)$$

The variable ξ denotes a random number uniformly generated in $[0, 1]$. Note that it holds $v_{t-1}^i - 1 \leq \mu(v_{t-1}^{i+1}, \delta_{t-1}^i) \leq v_t^i \leq v_{t-1}^i + 1$. Hence, acceleration and braking capabilities are limited, and a car changes its speed by at most ± 1.

2.1 Collision Free Driving

We will now determine the values of $\mu(v_t^{i+1}, \delta_t^i)$ for the mNaSch which ensure that there are no collisions between any two cars. (Valid initial configurations for open and periodic boundaries are given in appendix A.) We need to distinguish between two cases:

1. The vehicle i does not have a predecessor. This is only possible in an open system. In this case the car's speed is only limited by $v_{\max}$, the maximum technical speed of the car. (We assume here $v_{\max} = 6$.)

2. The vehicle i does have a predecessor. That means that there is another car driving ahead of vehicle i. This is always the case in a closed system. The maximum possible speed of car i at time t depends on (i) the speed of its predecessor and (ii) the distance to its predecessor at time $t-1$. This is captured by the function $\mu(v_t^{i+1}, \delta_t^i)$ defined as follows:
$$\mu(v_t^{i+1}, \delta_t^i) = \min\left\{ \left\lfloor \frac{1}{2}\sqrt{8\delta_t^i - 7 + 4v_t^{i+1}\left(v_t^{i+1} - 1\right)} - \frac{1}{2} \right\rfloor ;\ v_{\max} \right\}, \quad (2)$$
where $\lfloor\cdot\rfloor$ denotes the floor function and δ_t^i the distance between car i and its predecessor $i+1$ at time t.
The values resulting from Eq. (2) are given in table 1 for various combinations of a vehicle's headway and speed.

(To some extent this approach is comparable to the work of Emmerich and Rank [6], who investigated an update mechanism which takes into account both a vehicle's space gap and its speed as well. By ignoring the leading vehicle's speed this mechanism could not avoid collisions.)

Table 1. The values of the function $\mu(v_t^{i+1}, \delta_t^i)$ for $v_{\max} = 6$.

v_t^{i+1} / δ_t^i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	18	20	21	≥ 22
0	0	1	1	2	2	2	3	3	3	3	4	4	4	4	4	5	5	5	5	5	5	6
1	0	1	1	2	2	2	3	3	3	3	4	4	4	4	4	5	5	5	5	5	5	6
2	1	1	2	2	2	3	3	3	3	4	4	4	4	4	5	5	5	5	5	5	6	6
3	2	2	2	3	3	3	3	4	4	4	4	4	5	5	5	5	5	5	6	6	6	6
4	3	3	3	3	4	4	4	4	4	5	5	5	5	5	5	6	6	6	6	6	6	6
5	4	4	4	4	4	5	5	5	5	5	5	6	6	6	6	6	6	6	6	6	6	6
6	5	5	5	5	5	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6

Theorem 1. *It holds for all times ($\forall t \in \mathbb{N}$) that for any two cars i, j with $i < j$: $x_t^i < x_t^j$. This means that there are no collisions at any time.*

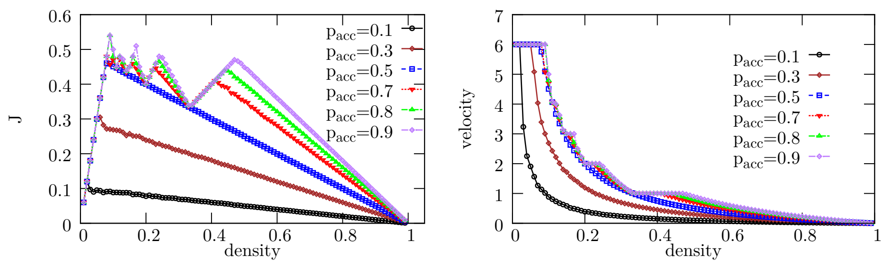
Proof. From table 1 we see that it holds
$$\mu(v_{t-1}^{i+1}, \delta_{t-1}^i) < \delta_{t-1}^i + \max\{v_{t-1}^{i+1} - 1, 0\} \quad (3)$$
and therefore (using $\delta_{t-1}^i = x_{t-1}^{i+1} - x_{t-1}^i$)
$$x_t^i \leq x_{t-1}^i + \mu(v_{t-1}^{i+1}, \delta_{t-1}^i) < x_{t-1}^{i+1} + \max\{v_{t-1}^{i+1} - 1, 0\} \leq x_t^{i+1}, \quad (4)$$
where we assume without loss of generality that $x_t^i < x_t^{i+1} \forall t \in \mathbb{N}$. Therefore, there are no collisions of any two cars for all times t.

Theorem 2. *By the definition of the function $\mu(v, \delta)$, the braking capabilities of the cars are limited. This means that the following inequality holds:*
$$\mu(v_t^{i+1}, \delta_t^i) \geq \mu(v_{t-1}^{i+1}, \delta_{t-1}^i) - 1 \ \forall t \in \mathbb{N}. \quad (5)$$

Proof. The theorem follows directly from the definition of $\mu(v_t^{i+1}, \delta_t^i)$ as shown in table 1.

3 Results

To begin our analysis we will present fundamental diagrams for different values of p_{acc}. For the simulation we used a road length of $L = 10^4$ sites and averages over $T = L$ time steps after a relaxation time of $10T$. Densities $0.01 \leq \rho \leq 1$ were simulated in steps of 0.01 for several values of p_{acc}. The fundamental diagrams are shown in figure 1(a) and the corresponding average speeds in 1(b).



(a) (b)

Fig. 1. For a periodic boundaries: (a) Fundamental diagram for different values of p_{acc}. (b) Corresponding average speeds.

Similar to the NaSch, the traffic flow increases rapidly up to a critical density. The reason is that for smaller densities all vehicles can accelerate to the maximal speed $v_{\max}$. Therefore, the flow rate J is given by $J = \rho \cdot v_{\max}$.

Unlike in the NaSch, traffic flow is not strictly monotonically decreasing for larger densities. The reason is the major difference between the NaSch and the mNaSch: as opposed to the NaSch, the mNaSch converges to stable states where all vehicles move with the same speed v or with two different speeds v and $v - 1$. We will refer to the first case as "speed-synchronized flow". When varying the system's density, the latter case can be regarded as a transition state between two speed-synchronized flows with speeds v and $v - 1$.

We use the term "speed-synchronized flow" to make clear that these phases are not necessarily identical with Kerner's three-phase traffic theory: In early investigations (e.g., [7]) the synchronized phase was identified as phase where vehicles travel with nearly identical speeds (even in different lanes) considerably below their maximum speed. In this sense, our speed-synchronized flow could be identified as Kerner's synchronized flow. Yet, more recent studies (summarized in [8]) have revealed a more complicated structure of synchronized flow.

As an example, figures 2(a) and 2(b) show the synchronization in a randomly initialized system with $\rho = 0.22$, $p_{\text{acc}} = 0.9$ and $L = 10^4$. Here the system converges to a stable state of speed-synchronized traffic flow where all cars have a speed of $v = 2$. Thereupon, $v^{\text{sf}}_{\rho,p_{\text{acc}}}$ refers to the speed of a stable state of speed-synchronized flow with given ρ and p_{acc}.

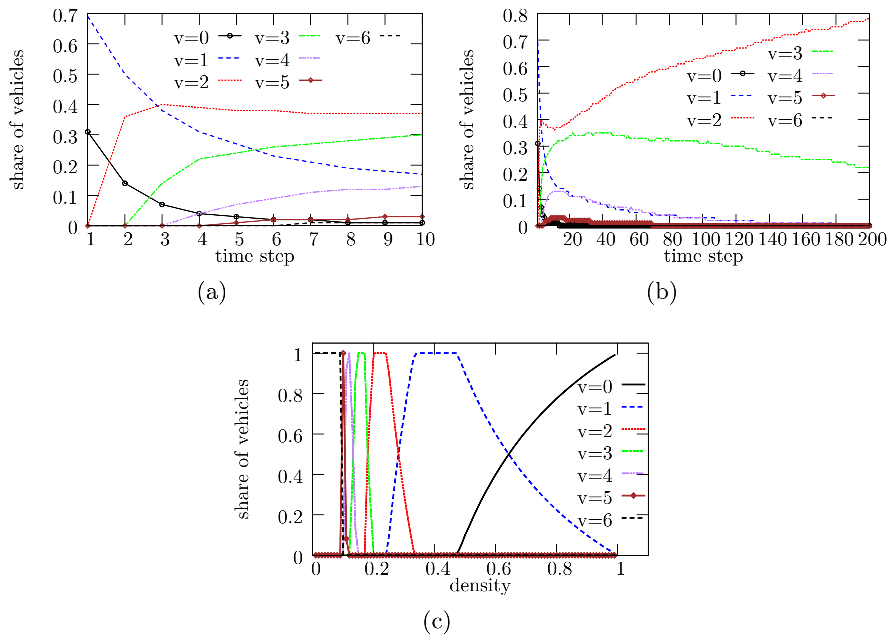


Fig. 2. For periodic boundary conditions one can see (a) the convergence to a stable state with speed-synchronized flow; time steps 1 to 10; $\rho = 0.22$, $p_{\text{acc}} = 0.9$ and $L = 10^4$. Each curve shows the percentage of cars with a certain speed. During the first few time steps cars frequently change their speeds. (b) After 200 time steps nearly all vehicles travel at a speed of $v = 2$ or $v = 3$. (c) The share of vehicles traveling at a given speed after a sufficient relaxation time (10^5 timesteps) for all densities with $p_{\text{acc}} = 0.9$.

We will now examine the relationship between ρ and $v^{\text{sf}}_{\rho,p_{\text{acc}}}$. We chose $p_{\text{acc}} = 0.9$ for our analysis because, with this value, the non-monotonical decrease for densities $\rho > \rho_{p_{\text{acc}}}$ can be seen particularly well in figure 1(a).

Figure 2(c) shows the share of vehicles with a given speed after 10^4 time steps for densities $\rho \in [0.01, 1]$ in steps of 0.01. Note that stable states of speed-synchronized flow with the same $v^{\text{sf}}_{\rho,0.9}$ are connected. We can therefore refer to a region of speed-synchronized flow when we mean a subset of [0.01,1] for which $v^{\text{sf}}_{\rho,0.9}$ is identical. Furthermore, $v^{\text{sf}}_{\rho,0.9}$ is monotonically decreasing with ρ. This is evident, for a higher density implies smaller distances between the cars in a stable state. Obviously traffic flow in a region of speed-synchronized flow is increasing with ρ. This is an important finding as it explains why the fundamental diagram is, in contrast to the fundamental diagram for the NaSch, not monotonically decreasing for $\rho > \rho_{p_{\text{acc}}}$.

4 Conclusion and Perspectives

The focus of this work was laid on a comparison of the Nagel Schreckenberg model and our modified version (mNaSch). We showed by means of simulations that the NaSch converges to either a steady state of speed-synchronized flow or a steady state with only two different speeds. We regard the latter case as a transition state between two states of speed-synchronized flow. We found differences between the fundamental diagrams of the two models. The fundamental diagrams of the mNaSch model have been shown to be more complex than those of the NaSch. Traffic flow is not simply increasing until a certain density is reached and decreases then, but it is moving in waves with the peak-values decreasing in density. The two principal differences between the NaSch and the mNaSch are that (i) in the mNaSch braking capabilities are limited and (ii) vehicles accelerate with a certain probability whereas in the NaSch vehicles decelerate at random.

A Initial Conditions Guaranteeing Collision Free Driving

The proof of collision free driving (section 2.1) requires that the road's previous configuration was free of collisions as well. Therefore, we present valid initial configurations for both open and periodic boundaries. First, the case of periodic boundaries: initially, N vehicles are randomly set on the road, and the initial speed of each vehicle is 0. Consequently, it holds that $x^i \neq x^j$ for $i \neq j$ and $\min(\delta^i) \geq 1 \; \forall i$. For all later times t, it follows $x^i_t = (x^i_{t-1} + v^i_t) \mod L$.

An open system represents a bottleneck situation where each car passes through the road only once. New cars enter the road via the left boundary, which requires that the leftmost site ($x = 1$) is empty. In this case a new car labeled k can be inserted with speed $v^k_t = \min\{v_{\text{in}}, \mu(v^{k+1}_{t-1}, x^{k+1}_{t-1} - 1)\}$ at position $x^k_t = 1 + v^k_t$, where $v_{\text{in}} = 2$ denotes the maximum speed of inserted cars. Afterwards, we apply the rules of motion to the remaining cars (i.e., all but the newly inserted one) and obtain the road's configuration at time t.

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